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Multi-photon Absorption Cross Section for a Maxwellian Electron Gas in a Uniform Magnetic Field

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Multi-photon absorption rate into a Maxwellian electron gas in a uniform magnetic field is calculated. The theory predicts that as the intensity of the applied photon gets larger, the multiphoton absorption process is considerably suppressed, thereby total absorption rate is saturated. As a simple illustration of this theory, single-photon absorption coefficient is calculated and compared with other theoretical results.

I. INTRODUCTION

In this paper we explore the nonlinear behavior of cyclotron absorption. A quantum mechanical method is adopted because of its convenience in dealing with the nonlinear problem. Firstly we calculate the transition probability per unit time for multi-photon absorption, and formulate the multi-photon absorption cross section.

Cyclotron(sometimes called synchrotron) radiation is an interaction between electrons and electromagnetic field. This is a process in which an electron, continuously accelerating in a magnetic field, emits or absorbs photons. In the magnetic confinement approach to controlled thermonuclear fusion, cyclotron absorption and emission will play an important role[1].

Theoretically, it can be shown, based on classical orbit theory, that the net power delivered to the electron is proportional to the square of the amplitude of the electro-magnetic radiation, that is, the intensity of the photon field[2]. It has been verified experimentally that up to a certain level of intensity, the absorption is proportional to the intensity of the injected wave, thus heating plasmas almost linearly. But, it is expected that as the intensity of ECRH (Electron Cyclotron Resonance Heating) increases, there may appear nonlinear effects which would result in a cut-off of the net absorption coefficient.

In order to deal such a nonlinear problem, we adopt a non-classical approach for the description of the electron, magnetic field, and electromagnetic wave system. In a static magnetic field, the electron's motion perpendicular to the magnetic field can be reduced to the equation of a harmonic oscillator. The electron's energy is therefore restricted to any one of the discrete energy states

$$E=\hbar\,\Omega_e\,(n+\tfrac{1}{2})+\frac{\hbar^2K^2}{2m} \tag{1}$$

where n is the quantum number of harmonic oscillator in the plane perpendicular to the magnetic field, Ωe is the electron cyclotron frequency $\left(=\frac{eB}{mc}\right)$, and $\hbar K$ is the momentum of electron in the direction of the magnetic field. Hence, when such an electron is in the presence of an electromagnetic wave (photon field), the electron may undergo transitions to higher energy states via photon absorption[3], provided that the wave frequency equals the cyclotron frequency, Ωe. It should be noted that the harmonic oscillator can jump into any level via interaction with the electromagnetic wave so that multi-photon absorption and emission are possible.

We are to obtain the transition probability to go from one state to another state with the absorption of arbitrary number of photons, and then we will explore some characteristics of nonlinear absorption.

The system consists of electrons gyrating around the magnetic field, and incoming photons.

The Hamiltonian for the system is taken to be

$$H=\frac{(\underline{P}-\frac{e}{c}\underline{A}^{r}-\frac{e}{c}\underline{A}^{e})^{2}}{2m}+H^{R} \tag{2}$$

where $\underline{P}$ is the momentum of the electron.
$\underline{P}-(e/c)\underline{A}^{e}$ is denoted as Q and the quadratic term in the vector potential $\underline{A}^{r}$ can be neglected for the emission and absorption problem. Then the Hamiltonian becomes

$$H=\frac{Q^{2}}{2m}+H^{PR}+H^{ER}+H^{R} \tag{3}$$

where $H^{R}=\hbar w a^{+}a$, $H^{PR}=-(e/mc)\underline{A}^{r}\cdot\underline{P}$, $H^{ER}=(e^{2}/mc^{2})\underline{A}^{r}\cdot\underline{A}^{e}$.
$\underline{A}^{e}$ is the vector potential of the applied magnetic field, and $\underline{A}^{r}$ is the radiation vector potential and is written as;

$$\underline{A}^{r}=c\left(\frac{2\pi\hbar}{L^{3}\omega}\right)^{\frac{1}{2}}e^{-i\underline{k}\cdot\underline{r}}\left\{a^{+}\underline{\epsilon}_{\lambda}+a\,\underline{\epsilon}_{\lambda}(-\underline{k})\right\} \tag{4}$$

The operators, a and a^{+}, are the conventional annihilation and creation operators of quantum electrodynamics[4], and L^{3} is the volume of the quantization box. The electromagnetic field is assumed to be pure, hence only one frequency (ω) and polarization $\underline{\epsilon}_{\lambda}$ appears in the expression for the vector potential $\underline{A}^{r}$.

Eigenstates may be displayed as $|\eta nKl>$. Here, the states $|\eta>$ and $|nKl>$ represent the free radiation field and the electron state respectively, where n and K are defined in eq.(1), l is the angular quantum number which designates the location of the guiding center. The following equation illustrates the energy of the electron,

$$\frac{Q^{2}}{2m}|nKl>=\left\{\hbar\Omega_{e}(n+\frac{1}{2})+\frac{\hbar^{2}K^{2}}{2m}\right\}|nKl>. \tag{5}$$

II. TRANSITION PROBABILITY PER UNIT TIME

In order to obtain the probability per unit length for a photon to be absorbed in an arbitrary number , we begin the analysis by considering the transition probability per unit time from the eigenstates $|\eta nKl>$ to $|\eta'n'K'l'>$ in general. Thus the quantity to be computed first is

$$T_{\eta'n'K'l',\ \eta nKl}=\frac{1}{t}<\eta'n'K'l'|U|\eta nKl>|^{2} \tag{6}$$

where U is the time evolution operator, $U=e^{-iHt/\hbar}$, and in a differential form, we can write

$$\frac{\partial U}{\partial t}=-\frac{i}{\hbar}HU. \tag{7}$$

We can decompose the operator U,

$$U=U^{R}U^{Q}\chi,$$

where

$$U^{R}=e^{-iH^{R}t/\hbar-iH^{PR}t/\hbar}$$

$$U^{Q}=e^{-iQ^{2}t/2m\hbar}. \tag{8}$$

Substituting eq.(8) into eq.(7) we obtain

$$U^{R}U^{Q}\frac{\partial\chi}{\partial t}=-\frac{i}{\hbar}H^{ER}U^{R}U^{Q}\chi. \tag{9}$$

Thus we can write

$$\frac{\partial\chi}{\partial t}=-\frac{i}{\hbar}H^{ER}(t)\chi, \tag{10}$$

where

$$H^{ER}(t) = U^{R^+} U^{Q^+} H^{ER} U^Q U^R . \tag{11}$$

We rewrite eq.(10) as an integral equation,

$$\chi(t) = 1 - \frac{i}{\hbar}\int_0^t dt' \, H^{ER}(t')\,\chi(t') . \tag{12}$$

In first iteration eq.(12) leads to

$$\chi(t) = 1 - \frac{i}{\hbar}\int_0^t dt' \, H^{ER}(t'). \tag{13}$$

Since the operation of U^R and U^Q on the state function does not make any contributions to the absolute value of the matrix element in eq.(6), the transition probability per unit time can be expressed as

$$T_{\eta' n' K' l',\, \eta n K l} = \frac{1}{t}\,|<\eta' n' K' l' \,|\, \chi \,|\, \eta n K l>|^2 . \tag{14}$$

Substituting χ into eq.(14) we obtain

$$\langle \eta' n' K' l' \,|\, \chi \,|\, \eta n K l \rangle$$

$$\simeq -\frac{i}{\hbar}\int_0^t dt' \,\langle \eta' n' K' l' \,|U^{R^+} U^{Q^+} H^{ER} U^Q U^R |\, \eta n K l \rangle$$

$$= -\frac{i}{\hbar}\int_0^t dt' e^{i(E'-E)t'/\hbar}$$

$$\langle \eta' n' K' l' \,|\, U^{R^+} H^{ER} U^R \,|\, \eta n K l \rangle \tag{15}$$

where

$$U^Q |\, nKl > = e^{-iEt/\hbar} |\, nKl >$$

and

$$E = \hbar\,\Omega_e (n + \frac{1}{2}) + \frac{\hbar^2 K^2}{2m} .$$

Next we proceed to evaluate the matrix element $<\eta' n' K' l' \,|U^{R^+} H^{ER} U^R \,|\, \eta n K l>$. For the uniform magnetic field in the z direction,

$$\underline{A}^e = \frac{1}{2}\underline{r} \times \underline{B} . \tag{16}$$

Then we can write

$$H^{ER} = \frac{e^2}{mc^2}\,\underline{A}^r \cdot \underline{A}^e = \frac{e^2}{2mc}\left(\frac{2\pi\hbar}{L^3\omega}\right)^{\frac{1}{2}}$$

$$e^{-i\underline{k}\cdot\underline{r}}\;\underline{r}\cdot(\underline{B}\times\underline{\epsilon})(a^+ + a). \tag{17}$$

Therefore the matrix element can be written as

$$\langle \eta' n' K' l' \,|U^{R^+} H^{ER} U^R \,|\, \eta n K l \rangle$$

$$\simeq \frac{e^2}{2mc}\left(\frac{2\pi\hbar}{L^3\omega}\right)^{\frac{1}{2}} (\underline{B}\times\underline{\epsilon})\cdot\langle \eta' n' K' l' \,|\, U^{R^+} e^{-i\underline{k}\cdot\underline{r}}$$

$$\underline{r}\,(a^+ + a) U^R |\, \eta n K l \rangle . \tag{18}$$

The matrix element in eq.(18) can be decomposed into two parts, the electron part and the photon part: the electron part results in

$$\langle n' K' l' | e^{-i\underline{k}\cdot\underline{r}}\,\underline{r} \,|\, nKl \rangle = i\,\delta_{KR}(K-K')\,\underline{\nabla}_k I_{n',n} \tag{19}$$

where

$$I_{n',n} = \langle n' l' | e^{-i\underline{k}\cdot\underline{r}} |\, nl \rangle \tag{20}$$

and the photon part results in

$$\langle \eta' \,|\, U^{R^+} (a^+ + a)\, U^R \,|\, \eta \rangle$$

$$= \sum_{\eta''} e^{-i\omega t} \sqrt{\eta'' + 1}\; e^{-\frac{x+x'}{2}} \alpha^m (-\beta')^{m_2}$$

$$\sqrt{\frac{\eta'!}{(\eta' + m_2)!}}\, L_{\eta'}^{m_2}(x')\, \sqrt{\frac{\eta!}{(\eta + m)!}}\, L_\eta^m$$

where the derivation is found in Appendix A and B. The transition probability of the system which absorbs m photons can be written as;

$$T_{\eta - m\; n' K' l',\; \eta n k K l}$$

$$= \frac{\pi e^4 \eta}{m^2 c^2 L^3 \omega}\left[(\underline{B}\times\underline{\epsilon})\cdot \underline{\nabla}_k I_{n',n}\right]^2$$

$$\delta_{KR}(K-K') J_m^2(\sqrt{\eta}\,Y)\,\delta_D(E' - E - m\hbar\omega), \tag{21}$$

where $I_{n',n}$ has been worked out by Parzen[3] and has been shown to be [3,5]

$$I_{n',n}=\sqrt{\frac{n'!}{n!}}e^{\alpha^2/2}\left(-\frac{\alpha^2}{2}\right)^{\lambda/2}L^{\lambda}_{n'}\left(\frac{\alpha^2}{2}\right)$$

$$\alpha=kb\sin\theta,\quad b^2=\frac{\hbar c}{eB},\quad \lambda=n-n'$$

and θ is the angle between the photon vector and z-axis. Let us denote $[(\underline{B}\times\underline{\epsilon})\cdot\underline{\nabla}_k I_{n',n}]^2$ as W, and note that $I_{n',n}$ has k_x and k_y dependence. Thus W can be written as

$$W=[(\underline{B}\times\underline{\epsilon})_x\frac{\partial}{\partial k_x}I_{n',n}(k_x,k_y)+(\underline{B}\times\underline{\epsilon})_y\frac{\partial}{\partial k_y}I_{n',n}(k_x,k_y)]^2$$

where

$$I_{n',n}=\sqrt{\frac{n'!}{n!}}\left(-\frac{b^2}{2}\right)^{\lambda/2}e^{-\frac{b^2}{2}(k_x^2+k_y^2)}(k_x^2+k_y^2)^{\lambda/2}\quad L^{\lambda}_{n'}\left(\frac{b^2}{2}(k_x^2+k_y^2)\right)$$

$$\frac{\partial I_{n',n}}{\partial k_x}=\sqrt{\frac{n'!}{n!}}\left(-\frac{b^2}{2}\right)^{\lambda/2}\Big\{-k_xb^2e^{-\frac{b^2}{2}(k_x^2+k_y^2)}(k_x^2+k_y^2)^{\lambda/2}L^{\lambda}_{n'}\left(\frac{b^2}{2}(k_x^2+k_y^2)\right)+e^{-\frac{b^2}{2}(k_x^2+k_y^2)}\lambda k_x(k_x^2+k_y^2)^{\lambda/2-1}L^{\lambda}_{n'}\left(\frac{b^2}{2}(k_x^2+k_y^2)\right)+e^{-\frac{b^2}{2}(k_x^2+k_y^2)}(k_x^2+k_y^2)^{\lambda/2}\frac{\partial}{\partial k_x}L^{\lambda}_{n'}\left(\frac{b^2}{2}(k_x^2+k_y^2)\right)\Big\}$$

If we restrict our attention to the case in which the photon vector and the polarization vector are perpendicular to the magnetic field, then $(\underline{B}\times\underline{\epsilon})_x=0$ $(\underline{B}\times\underline{\epsilon})_y=B$, $k=k_y$. We can write W ;

$$W=[(\underline{B}\times\underline{\epsilon})\cdot\underline{\nabla}_k I_{n',n}]^2=[B\frac{\partial}{\partial k_y}I_{n',n}]^2$$
$$=\frac{B^2n'!}{n!}\left(-\frac{b^2}{2}\right)^{\lambda}\Big\{-b^2e^{-\frac{b^2k^2}{2}}k^{\lambda+1}L^{\lambda}_{n'}\left(\frac{b^2k^2}{2}\right)+e^{-\frac{k^2b^2}{2}}\lambda k^{\lambda-1}L^{\lambda}_{n'}\left(\frac{b^2k^2}{2}\right)+e^{-\frac{b^2k^2}{2}}k^{\lambda}\frac{\partial}{\partial k}L^{\lambda}_{n'}\left(\frac{b^2k^2}{2}\right)\Big\}.$$

It should be noted that the term $J_m^2(\sqrt{\eta}Y)$ in eq.(21) containing nonlinear effects is the same as the one for inverse bremsstrahlung[6], thus similar nonlinear behavior is expected.

III. MACROSCOPIC MULTI-PHOTON ABSORPTION CROSS SECTION

We proceed to get a macroscopic absorption cross section, a^m, by summing all the final states, and averaging the initial states over the probability of the occupancy. a^m is the probability per unit path length per photon for m photons to be absorbed by an electron gyrating around the magnetic field.

$$a^m=\sum_{\substack{n,n'\\K,K'\\ \eta}}T_{\eta-m\,n'K'l',\eta nKl}\,P_\eta P_n P_K N^E/\frac{\eta}{L^3}C$$
$$=\sum_{\lambda}\frac{\pi e^4N^E}{m^2c^3\omega}\,W\,\delta(-\lambda\hbar\Omega_e-m\hbar\omega)\int d^3v f(v)J_m^2(\sqrt{\eta}Y)\qquad(22)$$

where P_η, P_n, P_K are probabilities of occupancy of state $|\eta\rangle$, $|n\rangle$, and $|K\rangle$, and $f(v)$ is the Maxwellian distribution.

For ECRH, if the ordinary and extraordinary wave are used as the mode of propagation, the wave propagation vector will be perpendicular to the magnetic field. Therefore, the angle between the magnetic field B and the polarization can take any value between 0 and π $(0<\varphi<\pi)$. Actually in most of the earlier heating experiments, the microwave power for ECRH was injected perpendicularly with respect to the magnetic field. But, these launching geometries resulted in edge plasma heating in some cases.

For a geometry in which the polarization vector and the photon vector are perpendicular to the direction of magnetic field, eq.(22) can be written as follows:

$$a^n = \sum_\lambda \frac{\pi e^4 N^E}{m^2 c^3 \omega} W\left(\frac{m}{2\pi\theta_T}\right)^{\frac{3}{2}} \int v_\perp dv_\perp dv_\parallel e^{-mv_\perp^2/2\theta_T} e^{-mv_\parallel^2/2\theta_T}$$
$$J_n^2(\sqrt{\eta}Y)\,\delta(-\lambda\hbar\Omega_e - n\hbar\omega) \tag{23}$$

where it should be noted that $K=K'$ is assumed for computational convenience, and $\alpha\, \underline{\epsilon}\cdot\underline{K} = \sqrt{\eta}Y$. Finally we obtain

$$a^n = \frac{e^4 N^E}{2mc^3\omega\theta_T}\left(\frac{\hbar}{\alpha m}\right)^2 W\int dq\, q\, e^{-q^2/\xi^2} J_n^2(q) \tag{24}$$

where we have introduced the notations $q=\frac{\alpha m v_\perp}{\hbar}$, $\xi^2 = \frac{2m\theta_T\alpha^2}{\hbar^2}$ and θ_T is the electron temperature.

The argument of Bessel function in eq.(24) contains the term proportional to the intensity of the applied photon field. Thus, for very large q, the Bessel function goes to zero, that is, the multiphoton absorption cross section vanishes. Therefore, for intense photon field, multiphoton absorption process is considerably suppressed. It should be noted that as can be seen from eq. (24) the argument of the Bessel function is related not only to the intensity of the applied photon field but also to the velocity perpendicular to the magnetic field. This means that the saturation behavior depends not only upon the intensity of radiation but also upon the energy of the gyrating electrons. Therefore the saturation of absorption rate at high intensity level of the applied photon field can happen more easily in very hot fusion plasma.

IV. CONCLUSION

As a simple illustration of quantum mechanical theory of multi-photon absorption, we can formulate the single photon absorption coefficient which is called linear cyclotron absorption coefficient otherwise. Detailed calculation and comparison with other results are in Appendix C.

At low and moderate level of intensity of the photon field, both single-photon and multi-photon processes contribute to absorption of the applied wave into the medium. However, at intense photon field multi-photon absorption is considerably suppressed, thereby limiting the absorption rate to saturated level, and the linear relation between absorption rate and the intensity of the photon field does not hold any more.

This theory can apply to Electron Cyclotron Resonance Heating for plasma fusion. Then it is theoretically predicted that as the intensity of a heating source, e.g, microwave, gets larger, there exists saturation phenomena of absorption rate which reduces the efficiency of a heating source.

Appendix A

In this Appendix, the matrix element $\langle \eta' | U^{R^+} (a^+ + a) U^R | \eta \rangle$ is calculated.

As defined earlier U^R has the form

$$U^R(\underline{k}, t') = e^{-i\omega a^+ a t' + i\gamma\underline{\epsilon}\cdot\underline{K}(a^+ + a)t'} \tag{A.1}$$

where $\gamma = \left(\frac{2\pi e^2\hbar}{m^2 L^3 \omega}\right)^{\frac{1}{2}}$

If we equate $U^R(\underline{K}, t)$ to $e^{-i\omega a^+ a t'}\zeta(t')$, then

$$\zeta(t') = e^{\frac{\gamma}{\omega}\underline{\epsilon}\cdot\underline{k}((a^+ - a) - e^{i\omega t'}a^+ + e^{-i\omega t'}a)} \tag{A.2}$$

Thus the matrix element can be written as

$$\langle \eta' | U^{R^+}(a^+ + a) U^R | \eta \rangle = \langle \eta' | \zeta^+ e^{i\omega a^+ a t'} a^+ e^{-i\omega a^+ a t'} \zeta | \eta \rangle + \langle \eta' | \zeta^+ e^{i\omega a^+ a t'} a\, e^{-i\omega a^+ a t'} \zeta | \eta \rangle \tag{A.3}$$

Note that

$$e^{i\omega a^+ a t'} a\, e^{-i\omega a^+ a t'} = e^{-i\omega t'} a \tag{A.4}$$

Similarly,

$$e^{i\omega a^+at'}a^+e^{-i\omega a^+at'} = e^{i\omega t'}a \quad \text{(A.5)}$$

Therefore we obtain

$$\langle \eta' | U^{R+}(a^+ + a)U^R | \eta \rangle = e^{i\omega t'} \langle \eta' | \zeta^+ a^+ \zeta | \eta \rangle + e^{-i\omega t'} \langle \eta' | \zeta^+ a \zeta | \eta \rangle \quad \text{(A.6)}$$

Let us keep the latter one for mathematical simplicity because the former is the complex conjugate of the latter. The final task is to evaluate $\langle \eta' | \zeta^+ a \zeta | \eta \rangle$, which can be written as follows:

$$= \sum_{\eta''} \sqrt{\eta''+1} \langle \eta' | \zeta^+ | \eta'' \rangle \langle \eta''+1 | \zeta | \eta \rangle \quad \text{(A.7)}$$

Introducing α, β we rewrite eq.(A.2)

$$\zeta(t) = e^{\alpha a^+ + \beta a} \quad \text{(A.8)}$$

where $\alpha = \frac{\gamma}{\omega} \underline{\epsilon} \cdot \underline{K} (1 - e^{i\omega t})$

$$\beta = -\frac{\gamma}{\omega} \underline{\epsilon} \cdot \underline{K} (1 - e^{-i\omega t})$$

After tedious but straightforward manipulation we obtain

$$\langle \eta''+1 | \zeta | \eta \rangle = e^{-x/2} \alpha^m \sqrt{\frac{\eta!}{(\eta+m)!}} L_\eta^m(x) \quad \text{(A.9)}$$

where $m = \eta'' + 1 - \eta$, $x = -\alpha\beta$ and $L_\eta^m(x)$ is the associated Laguerre polynomial of degree m and order η. The term $\langle \eta' | \zeta^+ | \eta \rangle$ can be obtained by applying complex conjugate to eq. (A.9). The whole matrix element $\langle \eta' | \zeta^+ a \zeta | \eta \rangle$ now reads:

$$I = \langle \eta' | \zeta^+ a \zeta | \eta \rangle$$
$$= \sum_{\eta''} \sqrt{\eta''+1}\, e^{-\frac{x+x'}{2}} \alpha^m (-\beta')^{m_2}$$
$$\sqrt{\frac{\eta'!}{(\eta'+m_2)!}} L_{\eta'}^{m_2}(x') \sqrt{\frac{\eta!}{(\eta+m)!}} L_\eta^m(x) \quad \text{(A.10)}$$

where $x' = -\alpha'\beta'$, $m_2 = \eta'' - \eta$

Back to eq.(15), we obtain

$$\langle \eta' n' K' l' | \chi | \eta n K l \rangle$$
$$\simeq -\frac{i}{\hbar} \frac{e^2}{2mc} \left(\frac{2\pi\hbar}{L'\omega}\right)^{\frac{1}{2}} (\underline{B} \times \underline{\epsilon}) \cdot i\, \delta(K-K') (\nabla_k I_{n', n})$$
$$\int_0^t dt' e^{i(E'-E)t'/\hbar} I e^{-i\omega t'} \quad \text{(A.11)}$$

Let us denote the integral in eq.(A.11) as P

$$P = \int_0^t dt' e^{i(E'-E)t'/\hbar} e^{-i\omega t'} I \quad \text{(A.12)}$$

After some tedius manipulation (Details are in Appendix B), P is worked out to be

$$P = \left(\frac{\eta+m+1}{\pi}\right)^{\frac{1}{2}} J_m(\sqrt{\eta}\, Y) \int_0^t dt' e^{i(E'-E-m\hbar\omega)t'/\hbar} \quad \text{(A.13)}$$

where

$$Y = \frac{4\gamma}{\omega} \frac{|\underline{\epsilon} \cdot \underline{K}| |\underline{\epsilon} \cdot \underline{K}'|}{\sqrt{|\underline{\epsilon} \cdot \underline{K}|^2 + |\underline{\epsilon} \cdot \underline{K}'|^2}}$$

and $\underline{K}$ and $\underline{K}'$ are the wave vectors of an electron before and after the interaction.

Thus,

$$\langle \eta' n' K' l' | \chi | \eta n K l \rangle$$
$$\simeq \frac{e^2}{2mc\hbar} \left(\frac{2\pi\hbar}{L^3\omega}\right)^{\frac{1}{2}} (\underline{B} \times \underline{\epsilon}) \cdot \nabla_k I_{n', n} \delta(K-K') \left(\frac{\eta+1}{\pi}\right)^{\frac{1}{2}}$$
$$J_m(\sqrt{\eta}Y) \int_0^t dt' e^{i(E'-E-m\hbar\omega)t'/\hbar} \quad \text{(A.14)}$$

Appendix B
CALCULATION OF P

In this Appendix, detailed calculation of P is presented.

From eq.(A.12)

$$P = \int_0^{t'} dt' e^{i(E'-E)t'/\hbar} e^{-i\omega t} I \quad \text{(B.1)}$$

where

$$I=\sum_{\eta''}\sqrt{\eta''+1}\,e^{-\frac{x+x'}{2}}\alpha^m(-\beta')^{m_2}\sqrt{\frac{\eta'!}{(\eta'+m_2)!}}L_{\eta'}^{m_2}(x')$$

$$\sqrt{\frac{\eta!}{(\eta+m)!}}\,L_\eta^m(x)$$

For simplicity in evaluating Laguerre polynomial, we assume

$$\eta'\simeq\eta,\quad m\simeq m_2$$

However, keep the difference $m_2=m+(n-1)$ for $(-\beta')^{m_2}$ term, if we consider n photon absorption, then

$$\eta'=\eta-n$$

Using the relation

$$L_\eta^m(x')\,L_\eta^m(x)=\frac{\Gamma(1+m+\eta)}{\eta!}\sum_{s=0}^{\infty}\frac{L_{\eta-s}^{m+2s}(x+x')(xx')^s}{\Gamma(1+m+s)s!} \tag{B.2}$$

We can write

$$I=\sum_{\eta''}\sqrt{\eta''+1}\,e^{-\frac{x+x'}{2}}(-\beta')^{n-1}(-\alpha\beta')^m$$
$$\sum_{s=0}^{\infty}\frac{(xx')^s}{(m+s)!s!}L_{\eta-s}^{m+2s}(x+x') \tag{B.3}$$

If we put $m=\eta''-\eta$

$$I=\sum_m\sqrt{\eta+m+1}\,e^{-\frac{x+x'}{2}}(-\beta')^{n-1}(-\alpha\beta')^m$$
$$\sum_{s=0}^{\infty}\frac{(xx')^s}{(m+s)!s!}L_{\eta-s}^{m+2s}(x+x') \tag{B.4}$$

where

$$\alpha=\frac{r}{\omega}\underline{\epsilon}\cdot\underline{K}(1-e^{i\omega t})=-\frac{r}{\omega}\underline{\epsilon}\cdot\underline{K}e^{i\omega t/2}2i\sin\omega t/2$$

$$\beta=-\frac{r}{\omega}\underline{\epsilon}\cdot\underline{K}(1-e^{-i\omega t})=-\frac{r}{\omega}\underline{\epsilon}\cdot\underline{K}e^{-i\omega t/2}2i\sin\omega t/2$$

$$x'=4\left(\frac{r}{\omega}\right)^2|\underline{\epsilon}\cdot\underline{K}'|^2\sin^2\frac{\omega t}{2}$$

$$\alpha\beta'=-4\left(\frac{r}{\omega}\right)^2|\underline{\epsilon}\cdot\underline{K}||\underline{\epsilon}\cdot\underline{K}'|\sin^2\frac{\omega t}{2}$$

$$x+x'=4\left(\frac{r}{\omega}\right)^2\sin^2\frac{\omega t}{2}\left\{|\underline{\epsilon}\cdot\underline{K}|^2+|\underline{\epsilon}\cdot\underline{K}'|^2\right\} \tag{B.5}$$

Substituting eq.(B.4) into eq.(B.1) we obtain

$$P=\sum_m\sqrt{\eta+m+1}\sum_{s=0}^{\infty}\frac{1}{(m+s)!s!}$$
$$\int_0^\tau dt\,e^{i(E'-E)t/\hbar}e^{-in\omega t}e^{-\frac{x+x'}{2}}(x)$$
$$(-\alpha\beta')^m(xx')^sL_{\eta-s}^{m+2s}(x+x') \tag{B.6}$$

Using eq.(B.5) P can be represented as

$$P=\sum_m\sqrt{\eta+m+1}\sum_{s=0}^{\infty}\frac{1}{(m+s)!s!}\left\{4\left(\frac{r}{\omega}\right)^2|\underline{\epsilon}\cdot\underline{k}|\right.$$
$$\left.|\underline{\epsilon}\cdot\underline{k}'|\right\}^{m+2s}(x)\int_0^\tau dt\,e^{i(E'-E-n\hbar\omega)t/\hbar}$$
$$e^{-\frac{x'+x}{2}}\sin^2\frac{\omega t}{2}L_{\eta-s}^{m+2s}(x+x') \tag{B.7}$$

Substituting the approximate formula for $L_{\eta-s}^{m+2s}(x+x')$

$$L_{\eta-s}^{m+2s}(x+x')=\frac{1}{\sqrt{\pi}}e^{\frac{x'+x}{2}}(x+x')^{-\frac{1}{2}(m+2s)-\frac{1}{4}}$$
$$(\eta-s)^{\frac{1}{2}(m+2s)-\frac{1}{4}}\simeq\frac{1}{\sqrt{\pi}}e^{\frac{y}{2}\sin\frac{\omega t}{2}}$$
$$[y\sin^2\frac{\omega t}{2}]^{-\frac{1}{2}(m+2s)-\frac{1}{4}}(\eta-s)^{\frac{1}{2}(m+2s)-\frac{1}{4}}$$

where

$$y=4\left(\frac{r}{\omega}\right)^2\left\{|\underline{\epsilon}\cdot\underline{K}|^2+|\underline{\epsilon}\cdot\underline{K}'|^2\right\} \tag{B.8}$$

From eq.(B.7) and eq.(B.8)

$$P=\sum_m\sqrt{\frac{\eta+m+1}{\pi}}\sum_{s=0}^{\infty}\frac{1}{(m+s)!s!}\left\{4\left(\frac{r}{\omega}\right)^2|\underline{\epsilon}\cdot\underline{K}|\right.$$
$$\left.|\underline{\epsilon}\cdot\underline{K}'|y^{\frac{1}{2}}\right\}^{m+2s}(\eta-s)^{\frac{1}{2}(m+2s)}$$
$$\int_0^\tau dt\,e^{i(E'-E-n\hbar\omega)t/\hbar} \tag{B.9}$$

Recall that

$$J_m(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(m+s)!}\left(\frac{x}{2}\right)^{m+2s}$$

Then we can write

$$P=\sum_m \sqrt{\frac{\eta+m+1}{\pi}}\, J_m(\sqrt{\eta}\,Y)\int_0^\tau dt\, e^{i(E'-E-n\hbar\omega)t/\hbar} \tag{B.10}$$

where

$$Y=4\frac{\gamma}{\omega}\,\frac{|\epsilon\cdot K||\epsilon\cdot K'|}{\sqrt{|\epsilon\cdot K|^2+|\epsilon\cdot K'|^2}}$$

Although m can take any integer, we simply pick up $m=n$ alone for a qualitative analysis.
The eq.(B.10) becomes

$$P=\sqrt{\frac{\eta+n+1}{\pi}}\, J_n(\sqrt{\eta}\,Y)\int_0^\tau dt\, e^{i(E'-E-m\hbar\omega)t/\hbar} \tag{B.11}$$

In order to avoid confusion with the quantum number n, we consider m photon absorption, and then P becomes

$$P=\sqrt{\frac{\eta+m+1}{\pi}}\, J_m(\sqrt{\eta}\,Y)\int_0^\tau dt\, e^{i(E'-E-m\hbar\omega)t/\hbar} \tag{B.12}$$

which is stated in eq.(A.13).

Appendix C
CALCULATION OF LINEAR CYCLOTRON ABSORPTION RATE

In this Appendix, the derivation of linear cyclotron absorption rate is presented.

Transition probability expressed in eq.(6) can reduce in first order to:

$$T_{b',b}=\frac{2\pi}{\hbar}\,|<b'|V|b>|^2\,\delta(E_{b'}-E_b) \tag{C.1}$$

where V is the interaction part of the Hamiltonian, and $E_b\cdot E_b'$ are energies of the system before and after the reaction.
We note that from eq.(3) the interaction part of the Hamiltonian is $-\frac{e}{mc}\,\underline{A}^r\cdot\underline{Q}$

Firstly, we calculate the transition probability for one photon emission.

$$T^e=\frac{2\pi}{\hbar}\,|<\eta+1\; n'K'l'|\frac{e}{mc}\underline{A}^r\cdot\underline{Q}\,|\eta nKl>|^2\delta(E'-E)$$
$$=\frac{4\pi^2e^2(\eta+1)}{m^2cL^3k}\,\delta(K'+K_{/\!/}'-K)\,|<n'l'|e^{-i\underline{k}_\perp\cdot\underline{r}}\,\underline{\epsilon}_\lambda\cdot\underline{Q}\,|nl>|^2\,\delta(E'-E) \tag{C.2}$$

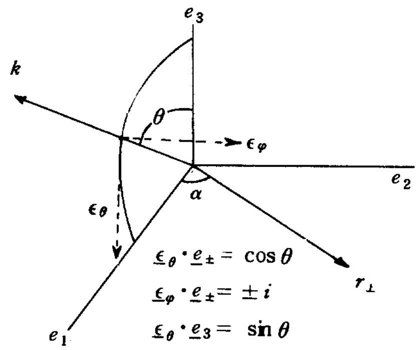
In this equation,

$$\underline{\epsilon}_\lambda\cdot\underline{Q}=\underline{\epsilon}_\lambda\cdot\underline{e}_1Q_1+\underline{\epsilon}_\lambda\cdot\underline{e}_2Q_2+\underline{\epsilon}_\lambda\cdot\underline{e}_3Q_3$$
$$=\underline{\epsilon}_\lambda\cdot\underline{e}_1\sqrt{\frac{\hbar m\Omega_e}{2}}(Q^++Q)+\underline{\epsilon}_\lambda\cdot\underline{e}_2\,i\sqrt{\frac{\hbar m\Omega_e}{2}}(Q^+-Q)+\underline{\epsilon}_\lambda\cdot\underline{e}_3\,\hbar K$$
$$=\sqrt{\frac{\hbar m\,\Omega_e}{2}}(\underline{\epsilon}_\lambda\cdot\underline{e}_+Q^++\underline{\epsilon}_\lambda\cdot\underline{e}_-Q)+\underline{\epsilon}_\lambda\cdot\underline{e}_3\hbar K$$

where we have introduced the vectors $\underline{e}_\pm=\underline{e}_1\pm i\underline{e}_2$ and defined $Q=\frac{1}{\sqrt{2m\hbar\Omega_e}}(Q_1'+iQ_2)$
Thus eq.(C.2) reads

$$T^e=\frac{2\pi^2e^2\hbar m\Omega_e(\eta+1)}{m^2cL^3k}\,\delta(K'+k_{/\!/}-K)\delta(E'-E)$$
$$|<n'l'|e^{-i\underline{k}_\perp\cdot\underline{r}}(\underline{\epsilon}_\lambda\cdot\underline{e}_+Q^++\underline{\epsilon}_\lambda\cdot\underline{e}_-Q+\sqrt{\frac{2\hbar}{m\Omega_e}}\,\underline{\epsilon}_\lambda\cdot\underline{e}_3K)|nl>|^2 \tag{C.3}$$

where $k_{/\!/}$ and $k_\perp$ are the components of the wave vector of the emitted photon parallel and perpendicular to the magnetic field respectively. We now choose a coordinate system as shown in the sketch. With this choice,



$$\underline{\epsilon}_\theta\cdot\underline{e}_\pm=\cos\theta$$
$$\underline{\epsilon}_\varphi\cdot\underline{e}_\pm=\pm i$$
$$\underline{\epsilon}_\theta\cdot\underline{e}_3=\sin\theta$$
$$\underline{\epsilon}_\varphi\cdot\underline{e}_3=0 \tag{C.4}$$

Note that

$$Q^{+}|n \quad 0>=\sqrt{n+1}\,|n+1 \quad 0>$$
$$Q\,|\,n \quad 0>=\sqrt{n}\,|\,n-1 \quad 0> \tag{C.5}$$

Thus the matrix element in eq.(C.3) is written as:

$$<|\,|>=\sqrt{\frac{2\hbar}{m\Omega_e}}K<n' \quad 0|\,e^{-ik_\perp r_\perp\cos\alpha}\underline{\epsilon}_\lambda\cdot\underline{e}_3|\,n \quad 0>$$
$$+\sqrt{n+1}<n' \quad 0|\,e^{-ik_\perp r_\perp\cos\alpha}\underline{\epsilon}_\lambda\cdot\underline{e}_+|n+1 \quad 0>$$
$$+\sqrt{n}<n' \quad 0|e^{-ik_\perp r_\perp\cos\alpha}\underline{\epsilon}_\lambda\cdot\underline{e}_-|\,n-1 \quad 0> \tag{C.6}$$

The term $<n' \, 0|e^{-i\underline{k}\cdot r}|\,n\,0>$ is denoted as $I_{n',n}$ in eq.(20). From Parzen's paper[3],

$$I_{n',n}=(-1)^{\lambda/2}e^{-\frac{\alpha^2}{\beta\sin\theta}}J_\lambda(\lambda\beta\sin\theta)$$

where $\lambda=n-n'$, $\alpha=kb\sin\theta$

$$\beta=\frac{v_\perp}{c}, \quad b^2=\frac{\hbar c}{eB}$$

Since we are concerned with φ-polarization, eq. (C.6) becomes

$$<|\,|>=\sqrt{n+1}\,i<n' \quad 0|\,e^{-ik_\perp r_\perp\cos\alpha}|n+1 \quad 0>$$
$$-\sqrt{n}\;i<n' \quad 0|e^{-ik_\perp r_\perp\cos\alpha}|n-1 \quad 0>$$
$$=i\left\{\sqrt{n}+1\,(-1)^{\lambda/2}e^{-\frac{\alpha^2}{\beta\sin\theta}}J_{\lambda+1}((\lambda+1)\beta\sin\theta)\right.$$
$$\left.-\sqrt{n}\,(-1)^{\lambda/2}e^{-\frac{\alpha^2}{\beta\sin\theta}}J_{\lambda-1}((\lambda-1)\beta\sin\theta)\right\} \tag{C.7}$$

Using the relation

$$J_{\lambda+1}(x)-J_{\lambda-1}(x)=2J_\lambda'(x)$$

We finally obtain the matrix element

$$<|\,|>=2i\sqrt{n}\,(-1)^{\lambda/2}e^{-\frac{\alpha^2}{\beta\sin\theta}}J_\lambda'(\lambda\beta\sin\theta) \tag{C.8}$$

Then eq.(C.3) is written as

$$T^{\varphi}=\frac{8\pi^2e^2\hbar(\eta+1)\Omega_e n}{mcL^3k}\delta(K'+k_\parallel-K)\,\delta(E'-E)$$
$$e^{-\frac{2\alpha^2}{\beta\sin\theta}}(x)\;[J_\lambda'(\lambda\beta\sin\theta)]^2 \tag{C.9}$$

where

$$E'=\hbar\Omega_e n'+\frac{\hbar^2K'^2}{2m}+\hbar w, \quad E=\hbar\Omega_e n+\frac{\hbar^2K^2}{2m}$$

Emission rate can be formulated by summing all final states and averaging the initial state.

$$\varepsilon^{\varphi}=\sum_\lambda\int d^3K'\int f(v)\,d^3v\;T^{\varphi}\,n^E/(\eta/L^3)$$
$$=\frac{4\pi^2e^2n^E}{w\hbar}\sum_\lambda\int d^3v\;f(v)\;v_\perp^2\,[J_\lambda'(\frac{\lambda v_\perp\sin\theta}{c})]^2$$
$$e^{-\frac{2\hbar\omega k\sin\theta}{m\Omega v_\perp}}$$
$$\delta\,(\lambda\Omega-\omega(1-\frac{v_\parallel}{c}+\frac{\hbar\omega\cos\theta}{2mc^2})] \tag{C.10}$$

where delta function comes from:

$$\int d^3K'\delta(K'+k_\parallel-K)\;\;\delta(E-E')$$
$$=\delta[\hbar\Omega_e\lambda+\frac{\hbar^2}{2m}(K^2-K^2+2Kk-k^2)-\hbar\omega]$$
$$=\frac{1}{\hbar}\delta[\lambda\Omega_e-\omega(1-\frac{v_\parallel}{c}+\frac{\hbar\omega\cos\theta}{2mc^2})] \tag{C.11}$$

and the exponent comes from:

$$-\frac{2\alpha^2}{\beta\sin\theta}=\frac{-2k^2b^2\sin^2\theta}{\beta\sin\theta}=\frac{-2\hbar^2\hbar c\sin\theta}{\frac{v_\perp}{c}\frac{eB}{m}m}$$
$$=\frac{-2k\hbar\omega\sin\theta}{v_\perp\Omega_e m} \tag{C.12}$$

Assuming $e^{-\frac{2\hbar\omega k\sin\theta}{v_\perp\Omega_e m}}\simeq 1$, $\frac{\hbar\omega\cos\theta}{2mc^2}\ll 1$ and thermal equilibrium,

$$\varepsilon^{\varphi}=\frac{4\pi^2e^2n^Ec}{\hbar\omega^2}\left(\frac{m}{2\pi k_BT}\right)^{3/2}$$

$$\sum_{\lambda} \int v_\perp \, dv_\perp \, dv_\parallel \, d\varphi \; e^{-m(v_\perp^2 + v_\parallel^2)/2k_BT} \; v_\perp^2$$
$$[J'_\lambda(\frac{\lambda v_\perp \sin\theta}{c})]^2 \; \delta(v_\parallel - c + \frac{\lambda \Omega_e c}{\omega}) \quad \text{(C.13)}$$

Substituting $x = \frac{mv_\perp^2}{2k_BT}$ we can write

$$\epsilon^\varphi = \frac{8\pi^2 e^2 c n^E}{\hbar \lambda^2 \Omega_e^2} \left(\frac{m}{2\pi k_BT}\right)^{3/2} \left(\frac{2k_BT}{m}\right)^2$$
$$\int_0^\infty x^3 dx \, e^{-x^2} [J'_\lambda(\lambda x \sqrt{\frac{2k_BT}{mc^2}})]^2 \quad \text{(C.14)}$$

Because of the smallness of $\frac{k_BT}{mc^2}$ we again resort to small argument approximation for the Bessel function, obtaining

$$\epsilon^\varphi = \frac{4\sqrt{\pi}\, n^E (mc^2)^2}{\hbar B^2} \left(\frac{2k_BT}{mc^2}\right)^{n-\frac{1}{2}}$$
$$\frac{(n+2)(n+1)\, n^{2n-3}}{2^{2n}(n-1)!} \quad \text{(C.15)}$$

for the n-th harmonic emission $(\omega = n\Omega_e)$ rate. This radiation is linearly polarized. The net absorption coefficient for this radiation is given by

$$\nu = \frac{\alpha^\varphi - \epsilon^\varphi}{c} = \frac{e^{\frac{\hbar\omega}{\theta}} - 1}{c}\, \epsilon^\varphi = \frac{\hbar\omega}{ck_BT}\, \epsilon^\varphi \quad \text{(C.16)}$$

which, using eq.(C.15) reduces to

$$\nu = 4\sqrt{\frac{\pi}{2}}\; \frac{en^E}{B} \left(\frac{k_BT}{mc^2}\right)^{n-\frac{3}{2}}$$
$$\frac{(n+2)(n+1)\, n^{2n-3}}{2^n (n-1)!} \; [\,\text{cm}^{-1}\,] \quad \text{(C.17)}$$

Note that for the first harmonic eq.(C.17) can be written as

$$\nu = 3\sqrt{\frac{1}{2\pi}}\; \frac{\omega_p^2}{\omega^2}\; \frac{\omega}{c} \left(\frac{k_BT}{mc^2}\right)^{-\frac{1}{2}} \quad \text{(C.18)}$$

Let us compare eq.(C.18) with the classical linear cyclotron absorption coefficient obtained by Akhiezer[7], which can be written as

$$\nu = \frac{1}{2}\sqrt{\frac{\pi}{2}}\; \frac{\omega_p^2}{\omega^2}\; \frac{\omega}{c}\; \frac{1+\cos^2\theta}{\cos\theta}\; e^{-z^2} \quad \text{(C.19)}$$

where

$$\beta_e = \left(\frac{k_BT}{mc^2}\right)^{1/2}$$
$$z = \frac{\omega - \Omega_e}{\sqrt{2}\,\beta_e\, \omega\cos\theta}$$

Note that with the small value of z, the quantum mechanical absorption coefficient is almost comparable to the classical absorption coefficient. We can compare eq.(C.18) with another classical linear result obtained by Manheimer[8,9]. In table(C.1) we list the comparison of linear cyclotron absorption coefficients by Chung, Akhiezer, and Manheimer. Chung's result was calculated quantum mechanically assuming the plåsma an ideal electron gas, thus neglecting the collective behavior of the plasma. Akhiezer's result was obtained for hot low density plasma. As can be seen from table(C.1), for low density plasma $(\omega_p \ll \omega)$, that is, when the collective behavior of plasma is neglected), $\left(2 - \frac{\omega_p^2}{\omega^2}\right)^{3/2}$ term in Man-

Table C.1 Comparison of Cyclotron Absorption Coefficients

Chung	$3\sqrt{\frac{1}{2\pi}}\; \frac{\omega_p^2}{\omega^2}\; \frac{\omega}{c} \left(\frac{k_BT}{mc^2}\right)^{-\frac{1}{2}}$
Akhiezer [7]	$\frac{1}{2}\sqrt{\frac{\pi}{2}}\; \frac{\omega_p^2}{\omega^2}\; \frac{\omega}{c} \left(\frac{k_BT}{mc^2}\right)^{-\frac{1}{2}} e^{-z^2}$
Manheimer [8]	$\frac{1}{8}\; \frac{\omega_p^2}{\omega^2}\; \frac{\omega}{c} \left(\frac{k_BT}{mc^2}\right)^2 \left(2 - \frac{\omega_p^2}{\omega^2}\right)^{3/2}$

* $z = \frac{\omega - \Omega_e}{\sqrt{2}\,\beta_e\,\omega\cos\theta}$, $\beta_e = \left(\frac{k_BT}{mc^2}\right)^{1/2}$

* All results are obtained for the first harmonic of extraordinary wave.

* Akhizer's result is obtained for hot low density plasma.

heimer's result is reduced to some constant, then all three results have the same dependence of the density and the magnetic field (ω_p and ω) Howeever, note that Chung and Akhizer have the temperature dependence of $T^{-(1/2)}$ whereas Manheimer has T^2 dependence.

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균일자기장 속의 맥스웰속도분포의 전자기체에서의 복수광자 흡수단면적

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(1987년 4월 13일 받음)

균일자기장 속에 있는 맥스웰속도분포를 가진 전자기체에서의 복수광자 흡수율을 계산하였다. 결과로서 입사하는 광자의 세기가 증가할수록 복수광자흡수 현상은 억제되고 전체흡수율이 포화되는 현상이 예측되었다. 이 양자역학적 모델의 타당성을 검증하기 위하여 단일광자흡수율을 계산하여 다른 이론치와 비교하였다.