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Influence of Ion Slip on the Magnetohydrodynamics in a Duct

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Ion slip is the dominant process by which the force $\underline{J}\times\underline{B}$ is transferred from the charge carrier to the gas as a whole. This force is exerted primarily on the electrons since they carry most of the current, but an electric field arises which communicates the force to the ions. As a result the ions and electrons together begin to move with a slip velocity, $\underline{u}_{is}$ relative to the whole fluid, which in turn gives rise to an opposing electric field (sometims called the Hall field), $\underline{E}_{Hall.}$[1]

$$\underline{E}_{Hall}=\frac{1}{c}\,\underline{u}_{is}\times\underline{B} \tag{1}$$

where

$$\underline{u}_{is}=(\underline{u}_i-\underline{u})_\perp=\frac{\underline{B}\times(\underline{u}_i-\underline{u})\times\underline{B}}{B^2},$$

$$\underline{u}=\frac{\Sigma_s m_s n_s \underline{u}_s}{\Sigma_s m_s n_s}\;(s=e,i,n),$$

m_s and n_s denote mass and density of species s, and $e,i,$ n denote electron, ion and neutrals, respectively.

In a recent paper,[2] W.B. Kunkel investigated the effect of neutral particles on the transport behavior of charged particles. He added an extra term in the generalized Ohm's law for partially ionized gases, to account for the relative motion of ions with respect to the fluid center-of-mass motion, that is, the ion slip effect.

The generalized Ohm's law can be written as follows;[2]

$$\underline{E}+\frac{\underline{u}\times\underline{B}}{c}=\eta\underline{j}+\left(\frac{\underline{j}\times\underline{B}}{c}-\nabla P_e+\underline{F}_e^{th}\right)/n_e e$$
$$+\frac{m_e\nu_{en}}{e}(\underline{u}_n-\underline{u}_i)-\frac{1}{c}\,\underline{u}_{is}\times\underline{B} \tag{2}$$

where η is the resistivity, $\quad\eta=\frac{m_e(\nu_{ei}+\nu_{en})}{e^2 n_e}$,

$\underline{F}_e^{th}$ is the thermal force,

$$\underline{u}_{is}=(1-X)(\underline{u}_i-\underline{u}_n)_\perp$$
$$=\frac{(1-X)^2\underline{j}\times\underline{B}}{m_i n_i\nu_{in}c}+\frac{X(1-X)\nabla_\perp P_n}{m_i n_i\nu_{in}}$$
$$-\frac{(1-X)^2\nabla_\perp(P_i+P_e)}{m_i n_i\nu_{in}}, \tag{3}$$

and X is the ion mass fraction,

$$X=\frac{m_i n_i}{\Sigma_s m_s n_s}\quad(s=e,\ i,\ n). \tag{4}$$

Sibstituting Eq. (3) into Eq. (2) we obtain

$$\underline{j}=\sigma\left(\underline{E}+\frac{\underline{u}\times\underline{B}}{c}\right)-\frac{\sigma}{n_e e}\left(\frac{\underline{j}\times\underline{B}}{c}-\nabla P_e+\underline{F}_e^{th}\right)$$
$$+\frac{\sigma(1-X)^2}{c^2 m_i n_i\nu_{in}}(\underline{j}\times\underline{B})\times\underline{B}+\frac{\sigma X(1-X)\nabla_\perp P_n\times\underline{B}}{m_i n_i\nu_{in}c}$$
$$-\frac{\sigma(1-X)^2\nabla_\perp(P_i+P_e)\times\underline{B}}{m_i n_i\nu_{in}c}. \tag{5}$$

Eq. (5) is the further development of Cowling's derivation of Ohm's law,

$$\underline{j}=\sigma\left(\underline{E}+\frac{\underline{u}\times\underline{B}}{c}\right)-\frac{\sigma}{n_e e}\frac{\underline{j}\times\underline{B}}{c}$$
$$+\frac{\Omega_e\Omega_i}{(\nu_{ei}+\nu_{en})\nu_{in}B^2}(\underline{j}\times\underline{B})\times\underline{B}, \tag{6}$$

including the ion mobility,[3] where Ω_e, Ω_i are the electron gyrofrequency and ion gyrofrequency respectively.

We can notice that Eq. (6) is the $X\to 0$ limiting case of Eq. (5), not having current driven by a pressure gradient.

Then the current density perpendicular to the magnetic field can be written as

$$\underline{j}_\perp=\frac{\sigma}{1+S}\left(\underline{E}_\perp+\frac{\underline{u}+\underline{B}}{c}\right)+\frac{X}{1-X}\frac{S}{1+S}\frac{c}{B}\nabla_\perp P_n$$

$$-\frac{S}{1+S}\frac{c}{B}\nabla_{\perp}(P_i+P_e) \quad (7)$$

where

$$S=\frac{(1-X)^2\Omega_e\Omega_i}{(\nu_{ei}+\nu_{en})\nu_{in}}$$

is called the ion slip coefficient.

Application of the generalized Ohm's law to the magnetohydrodynamic (MHD) generator h s appeared in the literature before[4,5]. A similar approach will be used here.

Fig. 1. shows a schematic illustration of the Magnetohydrodynamic (MHD) generator. We assume that $\underline{u}=u\, a_x$, $\underline{B}=B_0a_z$, $a>d$, and $L>d$, that is, the system is infinite in the x and y directions.

If the output is short circuited, ($E_\perp=0$), then

$$j=-\frac{\sigma}{c(1+S)}uB_0+\frac{X}{1-X}\frac{c}{B_0}\frac{S}{1+S}\frac{\partial P_n}{\partial y}$$
$$-\frac{c}{B_0}\frac{S}{1+S}\frac{\partial}{\partial y}(P_i+P_e). \quad (8)$$

We note th t the current density is decreased due to the ion slip.

With an open circuited output, ($j_\perp=0$), the device acts as a flow meter, in which the output voltage is

$$V=aE_\perp=a[\frac{uB_0}{c}-\frac{X}{1-X}\frac{cS}{\sigma B_0}\frac{\partial P_n}{\partial y}$$
$$+\frac{cS}{\sigma B_0}\frac{\partial}{\partial y}(P_i+P_e)]. \quad (9)$$

It should be noted that in Eqns. (8) and (9), the pressure gradient terms are small compared to the motional emf term, $\frac{(\underline{u}\times\underline{B})}{c}$, so for engineering application the pressure gradient terms can be ignored. However, for nonuniform plasma in an MHD generator duct, those terms should be considered.[6]

We write the momentum conservation equation including the viscosity,

$$\rho\frac{D\underline{u}}{Dt}=-\nabla P+\frac{\underline{j}\times\underline{B}}{c}+\frac{\eta_v}{3}\nabla(\nabla\cdot\underline{u})+\eta_v\nabla^2\underline{u} \quad (10)$$

or

$$(m_nn_n+m_in_i)\frac{D\underline{u}}{Dt}+\nabla(P_n+P_i+P_e)=$$
$$\frac{\underline{j}\times\underline{B}}{c}+\frac{\eta_v}{3}\nabla(\nabla\cdot\underline{u})+\eta_v\nabla^2\underline{u} \quad (11)$$

where η_v is the viscosity and $\rho\simeq m_nn_n+m_in_i$,

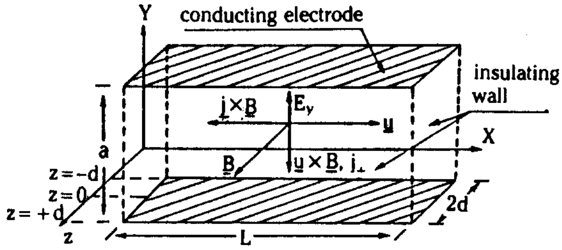

Fig. 1. Schematic diagram of a gas flow duct in an MHD generator.

$P=P_i+P_e+P_n$

If we assume a steady state, incompressible flow and neglect the convective term, Eq. (10) reduces to

$$\nabla P=\frac{1}{c}(\underline{j}\times\underline{B})+\eta_v\nabla^2\underline{u}. \quad (12)$$

Substituting the current density, Eq. (7), into Eq. (12) we obtain the three component equations in (12) as

$$\frac{\partial P}{\partial x}=\frac{\sigma B_0}{c(1+S)}[E_\perp-\frac{1}{c}B_0u_x(z)]$$
$$+\frac{X}{1-X}\frac{S}{1+S}\frac{\partial P_n}{\partial y}$$
$$-\frac{S}{1+S}\frac{\partial}{\partial y}(P_i+P_e)+\eta_v\frac{d^2u_x}{dz^2},$$

$$\frac{\partial P}{\partial y}=0,$$

$$\frac{\partial P}{\partial z}=-\frac{\sigma B_x}{c(1+S)}[E_\perp-\frac{1}{c}B_0u_x(z)]$$
$$-\frac{X}{1-X}\frac{B_x}{B_0}\frac{S}{1+S}\frac{\partial P_n}{\partial y}$$
$$+\frac{S}{1+S}\frac{B_x}{B_0}\frac{\partial}{\partial y}(P_i+P_e), \quad (13)$$

where B_x is the axial magnetic induction caused by the transport of magnetic field lines in the direction of flow due to fluid motion.

We can rewrite Eq. (13) as

$$\frac{d^2u_x}{dz^2}-\frac{\sigma B_0^2}{\eta_vc^2(1+S)}u_x=\frac{1}{\eta_v}\frac{\partial P}{\partial x}-\frac{\sigma E_\perp B_0}{\eta_vc(1+S)}$$
$$-\frac{1}{\eta_v}\frac{X}{1-X}\frac{S}{1+S}\frac{\partial P_n}{\partial y}+\frac{1}{\eta_v}\frac{S}{1+S}\frac{\partial}{\partial y}(P_i+P_e). \quad (14)$$

We consider here two representative cases; one is when we have a constant pressure gradient and stationary insulating walls (Fig. 1), the other is when we

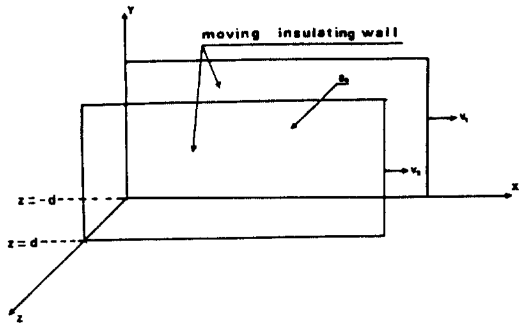

Fig. 2. Flow of viscous conducting fluid in a magnetic field between two insulating walls moving with different velocities.

have no pressure gradient in the x direction and insulating walls moving in the x direction (Fig. 2).

The flow velocity in the x direction, $u_x(z)$, and induced axial magnetic field $B_x(z)$ will be obtained for both cases.

CaseI: *Within a duct with stationary walls and constant pressure gradient in the gas flowing direction.* As noted before, for engineering application, the pressure gradient in the y direction can be neglected and we assume a constant pressure head ($P_o=-\frac{dP}{dX}$).Then Eq. (14) can be written as

$$\frac{d^2u_x}{dz^2}-\frac{\sigma B_o^2}{\eta_v c^2(1+S)}u_x(z)=-\frac{P_o}{\eta_v}-\frac{\sigma E_\perp B_o}{\eta_v c(1+S)}. \tag{15}$$

With the condition that the flow velocity must vanish at the walls ($z=\pm d$), the solution to Eq. (15) is

$$u_x(z)=\frac{(1+S)c^2P_o+\sigma cB_oE_\perp}{\sigma B_o^2}\left[1-\frac{\cosh(\frac{Mz}{d})}{\cosh M}\right], \tag{16}$$

where $M=\frac{B_o d}{c}[\frac{\sigma}{\eta_v(1+S)}]^{1/2}$ is the modified Hartmann number for partially ionized gases.

The average flow velocity is

$$\overline{u_x}=\frac{1}{2d}\int_{-d}^{d}u_x(z)\,dz=\frac{(1+S)c^2P_o+\sigma cB_oE_\perp}{\sigma B_o^2}\left[1-\frac{1}{M}\tanh M\right]. \tag{17}$$

From the solution, Eq. (16), we notice that the velocity profile flattens out and wall effects are localized as the flow becomes less viscous, or as the width of the duct increases. And we note that ion slip caused by the existence of neutrals increases the wall effect in the duct.

Also as can be noted from Eqs. (16) and (17), in the limit $M\gg 1$, the flow is determined entirely by the $\underline{E}\times\underline{B}$ drift.

The magnetic field $B_x(z)$ is determined by equation,[7]

$$\frac{dB_x}{dz}=\frac{4\pi}{c}J_y\simeq\frac{4\pi}{c}\frac{\sigma}{1+S}[E_\perp-\frac{1}{c}B_o u_x(z)]. \tag{18}$$

Thus, by integration of Eq. (18) we obtain

$$B_x(z)=\frac{4\pi}{c(1+S)}[\sigma E_\perp-\frac{cP_o(1+S)+\sigma B_oE_\perp}{B_o}]z-\frac{4\pi d}{c(1+S)M}\frac{cP_o(1+S)+\sigma B_oE_\perp}{B_o}\frac{\sinh\frac{Mz}{d}}{\cosh M} \tag{19}$$

CaseII: *Within a duct with no pressure gradient in the x direction and moving insulated walls.*

If we have no pressure gradient in the x direction and two insulating surfaces move with velocities v_1 and v_2 in the x direction, respectively (Fig. 2), the flow velocity equation in this case is

$$\frac{d^2u_x}{dz^2}-(\frac{M}{d})^2u_x(z)=-(\frac{M}{d})^2\frac{cE_\perp}{B_o}. \tag{20}$$

With the boundary conditions $u(d)=v_2$, $u(-d)=v_1$, we obtain the solution

$$u_x(z)=[\frac{v_1+v_2}{2}-\frac{cE_\perp}{B_o}]\frac{\cosh\frac{Mz}{d}}{\cosh M}+\frac{v_2-v_1}{2}\frac{\sinh\frac{Mz}{d}}{\sinh M}+\frac{cE_\perp}{B_o}. \tag{21}$$

The average flow velocity is

$$\bar{u}_x=\frac{1}{M}[\frac{v_1+v_2}{2}-\frac{cE_\perp}{B_o}]\tanh M+\frac{cE_\perp}{B_o}. \tag{22}$$

In the limit $B_o\to 0$, $M\to 0$, we obtain the standard laminar-flow result

$$\bar{u}_x(z)\simeq\frac{v_1+v_2}{2}+\frac{v_2-v_1}{2}(\frac{z}{d}). \tag{23}$$

In the other limit of $M\gg 1$, we approximate $u_x(z)$ as

$$u_x(z) \cong \left(v_2 - \frac{cE_\perp}{2B_o}\right) e^{\frac{M(z-d)}{d}} + \left(v_1 - \frac{cE_\perp}{2B_o}\right) e^{-\frac{M(z+d)}{d}} + \frac{cE_\perp}{B_o}. \quad (24)$$

Again for $M \to 0$, laminar flow occurs, and for $M \gg 1$ the flow is given by the $\underline{E} \times \underline{B}$ drift velocity.

The axial component of the magnetic induction is obtained as before,

$$B_x(z) = \frac{4\pi d}{c^2 M} \frac{\sigma B_o}{1+S} \left[\frac{v_2 - v_1}{2} \frac{\cosh \frac{Mz}{d}}{\sinh M} - \left(\frac{v_1 + v_2}{2} - \frac{cE_\perp}{B_o} \right) \frac{\sinh \frac{Mz}{d}}{\cosh M} \right]. \quad (25)$$

The results of this work can be stated as folows;

(1) The generalized Ohm's law, Eq. (4), was derived on the assumption that the electron gyrofrequency, Ω_e, is greater than the collision frequncy of electrons with neutrals, ν_{en}, and that neutral slip is small.

For an MHD generator duct employing a several kilo gauss magnetic field, these conditions are easily fulfilled.

(2) The ion slip caused by the existence of neutral particles decreases the current density and increases the wall effect in a gas flow duct.

(3) The formulae presented here can give a good estimate of the velocity and axial magnetic induction profile for various partially ionized gases used in an MHD generator duct with an arbitrary degree of ionization.

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이온슬립이 관내의 자기유체역학에 미치는 영향

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(1988년 3월 14일 받음)

자기유체발전관내의 중성입자에 기인한 이온슬립이 관내의 Hartmann flow 및 축방향자기유도에 미치는 영향을 두개의 대표적인 절연체벽 구조에 대하여 계산하였다. 그 결과로서 전류밀도는 감소하고 관내의 장벽효과는 증가하는 것으로 나타났다.