

Interaction of a Gyrating Free-Electron with a Single-Mode Quantized Electromagnetic Field

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A quantum mechanical description of the interaction of a gyrating free-electron in uniform magnetic field with a single-mode fully quantized electromagnetic field is presented. The interaction Hamiltonian including RWA terms are obtained for the linearly polarized and circularly polarized electromagnetic waves. And the normal mode frequency of the vibrations and eigenstates are calculated. Emission and absorption characteristics associated with cyclotron motion of an electron are also discussed.

§1. Introduction

Cyclotron radiation is a process in which an electron, continuously accelerating in a magnetic field, emits or absorbs photons.¹⁾ The motion of gyrating electron can be described in terms of the harmonic oscillation in the plane perpendicular to the applied uniform magnetic field. Thus the gyrating electron is called charged harmonic oscillator.²⁾ When a charged harmonic oscillator is exposed to a single mode electromagnetic wave (another harmonic oscillator), the coupling between two oscillator results in new resonance frequency (normal mode of vibrations).³⁾ We consider the interaction of a gyrating free-electron with a fully quantized electromagnetic field, which is proved to be important for dealing with quantum optics problems.⁴⁻⁷⁾

Theoretical description of the interaction of an electron (free or bound) with a single-mode quantized electromagnetic field is a valuable reference for analyzing the results of experiments in various areas. Intensive investigations have been focused on the understanding of the interaction of a two-level atomic system with a quantized field. The approximations used frequently in quantum optics calculation are the dipole approximation, the neglect of the diamagnetic term, and the rotating wave approximation (RWA).⁸⁻¹⁰⁾ The energy non-conserving RWA terms exhibit peculiar characteristics in some experiments, for instance, these terms give rise to a chaotic behavior in strong

electric field,^{11,12)} and Bloch-Siegert shift.^{13,14)}

In this study we focus our attention to the problem of the interaction of a free electron in a uniform magnetic field with a fully quantized electromagnetic field. The interaction Hamiltonian in this case has RWA terms. We obtain the interaction Hamiltonian for two representative polarization of the incident em wave. Then we investigate the effect of RWA terms on the normal mode frequency and the absorption rate of em wave into the electron gas.¹⁵⁻¹⁷⁾ For the calculation of the absorption rate we assume the electron gas has Maxwell-Boltzman energy distribution. In the magnetic confinement approach to controlled thermonuclear fusion, cyclotron absorption and emission play an important role. Besides, the underlying physics can be applied to the microwave discharge of plasma with the magnetic field, and to the microwave etching. The physics of the microwave interaction with plasma should be understood for the progress of VLSI processing, high current ion source, or other plasma processing.^{18,19)}

§2. Formulations

In a static magnetic field, the electron's motion perpendicular to the magnetic field can be reduced to the equation of a harmonic oscillator. The electron's energy is therefore restricted to any one of the discrete energy states

$$E_{nK} = \hbar\Omega_e \left(n + \frac{1}{2} \right) + \frac{\hbar^2 K^2}{2m}, \quad (1)$$

where n is the quantum number of harmonic oscillator, Ω_e is the electron cyclotron frequency ($=eB/mc$), and $\hbar K$ is the momentum of electron in the direction of the magnetic field. Hence, when such an electron is in the presence of an electromagnetic wave (photon field), the electron may undergo transitions to higher energy states via photon absorption, provided that the wave frequency matches the cyclotron frequency, Ω_e . The Hamiltonian for a single electron gyrating around the static magnetic field in the presence of a quantized electromagnetic field is

$$H = \frac{(\mathbf{p} - e/c\mathbf{A}_e - e/c\mathbf{A}_r)^2}{2m} + H_F, \quad (2)$$

where A_e is the vector potential due to the static magnetic field and A_r is the vector potential due to quantized em field, H_F is the Hamiltonian for the quantized em field. Equation (2) can be written as

$$H = H_0 + H_I + H_F \quad (3a)$$

where the Hamiltonian for the unperturbed electron is

$$H_0 = \frac{(\mathbf{p} - e/c\mathbf{A}_e)^2}{2m}. \quad (3b)$$

The Hamiltonian for the single-mode field is written as

$$H_F = \hbar\omega \left(a^+ a + \frac{1}{2} \right), \quad (3c)$$

where a and a^+ are the annihilation and creation operators of quantum electrodynamics. Equation (3c) can be written in terms of canonical variables Q and P ,

$$H_F = \frac{1}{2} (P^2 + \omega^2 Q^2). \quad (3d)$$

Finally, the interaction Hamiltonian is

$$H_I = -\frac{e}{mc} \mathbf{A}_r \cdot \left(\mathbf{p} - \frac{e}{c} \mathbf{A}_e \right) + \frac{e^2}{2mc^2} \mathbf{A}_r^2. \quad (3e)$$

The eigenstate of the Hamiltonian may be displayed as $|\eta_k, n, K, l\rangle$. Here the states $|\eta_k\rangle$ and $|n, K, l\rangle$ represent the quantized field and the electron state respectively, and n and K are defined in eq. (1), l is the angular momentum number which designates the location of the

guiding center. The eigenstates of H_0 is written as

$$H_0 |n, K, l\rangle = E_{nK} |n, K, l\rangle. \quad (4)$$

A linearly polarized single-mode electromagnetic field is usually written as;

$$\mathbf{A}_r(r, t) = \sum_{\lambda=1,2} c \left(\frac{2\pi\hbar}{V\omega} \right)^{1/2} \times e^{-ik \cdot r} [a^+ \boldsymbol{\varepsilon}_\lambda + a \boldsymbol{\varepsilon}_\lambda(-\mathbf{k})], \quad (5)$$

where $\boldsymbol{\varepsilon}_\lambda$ is a unit polarization vector and V is the volume of the quantization box. For a mode polarized in the x direction and in the electric dipole approximation, eq. (5) can be simplified as

$$\mathbf{A}_r(0, t) = c \left(\frac{2\pi\hbar}{V\omega} \right)^{1/2} \mathbf{e}_x (a^+ + a). \quad (6)$$

The interaction Hamiltonian H_I contains a diamagnetic term $e^2 A_r^2 / 2mc^2$ which has been usually neglected for emission and absorption problem. Substitution of eq. (5) into the diamagnetic term leads to²⁾

$$\frac{e^2 A_r^2}{2mc^2} = \hbar\gamma (a^2 + a^{+2} + 2a^+ a + 1) = 2\gamma\omega Q^2 \quad (7)$$

where

$$\gamma = \frac{e^2 \pi}{m\omega V}.$$

Thus, the Hamiltonian can be written as

$$H = \hbar\Omega_e \left(b^+ b + \frac{1}{2} \right) + \frac{1}{2} (P^2 + v^2 Q^2) - 2 \frac{e}{m} \left(\frac{\pi}{V} \right)^{1/2} \mathbf{e}_x \cdot \left(\mathbf{p} - \frac{e}{c} \mathbf{A}_e \right) \quad (8)$$

where

$$v^2 = \omega^2 + 4\gamma\omega,$$

$$b = \frac{1}{\sqrt{2m\hbar\Omega_e}} (m\Omega_e x + ip_x),$$

$$Q = \sqrt{\frac{\hbar}{2v}} (a + a^+).$$

Note that the diamagnetic term can be combined with the field Hamiltonian to provide a new single-mode field Hamiltonian with a slightly shifted frequency. The new field mode frequency ν can be reduced $\nu^2 = \omega_p^2 + k^2 c^2$

when the replacements of $\omega \rightarrow kc$ and $(1/V) \rightarrow n_e$ are made, where $\omega_p^2 = (4\pi n_e e^2/m)$, n_e is the electron density, and k is the wavenumber of electromagnetic wave. The frequency shift due to diamagnetic term is in analogy with the additive contribution of individual atoms of a gas to the gas's dielectric constant.

For a static magnetic field in the z -direction, we have

$$\mathbf{A}_e = -\frac{1}{2} \mathbf{r} \times \mathbf{B} = -\frac{yB}{2} \mathbf{e}_x + \frac{xB}{2} \mathbf{e}_y. \quad (9)$$

Thus the Hamiltonian of eq. (8) can be expressed as;

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m \Omega_e^2 r^2 + \frac{1}{2} (P^2 + v^2 Q^2) + H_1, \quad (10)$$

where the interaction Hamiltonian is written as

$$H_1 = -2 \frac{e}{m} \left[\frac{\pi}{V} \right]^{1/2} Q p_x - e \Omega_e \left(\frac{\pi}{V} \right)^{1/2} Q y. \quad (11)$$

The Hamilton's equation of motion is rewritten as a set of second-order differential equation for coupled harmonic oscillators;

$$\begin{pmatrix} \Omega_e^2 & 0 & 0 & 0 & 0 & 2c_1 \\ 0 & \Omega_e^2 & 0 & 0 & -c_1 \Omega_e & 0 \\ 0 & 0 & \Omega_e^2 & 0 & -2c_1 \Omega_e & 0 \\ 0 & 0 & 0 & \Omega_e^2 & 0 & -c_2 \\ 0 & 0 & -2c_1 & 0 & v^2 & 0 \\ 2c_2 \Omega_e & 0 & 0 & 0 & 0 & v^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \\ Q \\ P \end{pmatrix} = \lambda^2 \begin{pmatrix} x \\ y \\ p_x \\ p_y \\ Q \\ P \end{pmatrix} \quad (12)$$

where

$$c_1 = \frac{e}{m} \left[\frac{\pi}{V} \right]^{1/2}, \quad c_2 = e \Omega_e \left(\frac{\pi}{V} \right)^{1/2},$$

and we assume that the canonical variables have time dependence of $e^{i\lambda t}$. The normal mode frequency of the coupled harmonic oscillator can be given by solving the eigen-value equation of eq. (12). The eigen-values λ^2 are;

$$\lambda^2 = \frac{\Omega_e^2}{2} \pm \frac{\Omega_e^2 + v^2 \cdot [(\Omega_e^2 - v^2)^2 + 16(\Omega_e^2 e^2 \pi / m V)]^{1/2}}{2}.$$

This corresponds to the wave dispersion relation for the electromagnetic waves which propagate perpendicularly to the external magnetic field. However, the plasma wave theory gives;²⁰⁾

$$\lambda^2 = \frac{\omega_h^2 + v^2}{2} \pm \frac{[(\omega_h^2 - v^2)^2 + 4\Omega_e^2 \omega_p^2]^{1/2}}{2}$$

where

$$\omega_h^2 = \Omega_e^2 + \omega_p^2.$$

The difference can be explained as; the quan-

tum mechanical approach is basically single-particle theory, the condition $\Omega_e/\omega_p \gg 1$ is implied in this formalism. With the replacements of $\Omega_e \rightarrow \omega_h$, and $(1/V) \rightarrow n_e$, we can approach to the results obtained from plasma wave theory.

The expansion of the vector potential, polarized in the xy plane, in the circularly polarized representation is³⁾

$$\mathbf{A}_\tau(\mathbf{r}, t) = \sqrt{\frac{2\pi\hbar c}{Vk}} (e_- a_- + e_-^* a_-^\dagger + e_+ a_+ + e_+^* a_+^\dagger), \quad (13)$$

where the circularly polarized unit vector \mathbf{e}_\pm are related to the linearly polarized vector \mathbf{e}_x and \mathbf{e}_y according to

$$\mathbf{e}_+ = -\frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, \quad \mathbf{e}_- = \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}}, \quad (14)$$

and correspond, respectively, to the RHCP and LHCP unit vectors. The relation between the circularly polarized and the linearly polarized creation and annihilation operators

$$\begin{aligned}
 a_x &= -\frac{a_+ - a_-}{\sqrt{2}} & A_r(\mathbf{r}, t) &= \sqrt{\frac{2\pi\hbar c}{Vk}} [e_x(a_x + a_x^+) + e_y(a_y + a_y^+)]. \\
 a_y &= -i\frac{a_+ + a_-}{\sqrt{2}}, & & (15)
 \end{aligned}$$

have been used. Thus the eq. (13) can be rewritten in terms of linearly polarized operators.

This can also be written as

$$A_r(\mathbf{r}, t) = 2c \left(\frac{\pi}{V} \right)^{1/2} (Q_x e_x + Q_y e_y). \quad (17)$$

Thus the interaction Hamiltonian has form

$$H_1 = -2 \frac{e}{m} \left[\frac{\pi}{V} \right]^{1/2} Q_x p_x - e\Omega_e \left(\frac{\pi}{V} \right)^{1/2} Q_x y - 2 \frac{e}{m} \left[\frac{\pi}{V} \right]^{1/2} Q_y p_y + e\Omega_e \left(\frac{\pi}{V} \right)^{1/2} Q_y x. \quad (18)$$

The Hamiltonian consists of two parts;

$$H = H_0 + H_1$$

where

$$H_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m\Omega_e^2 r^2 + \frac{1}{2} (\mathbf{P}^2 + v^2 \mathbf{Q}^2). \quad (19)$$

The Hamilton's equation of motion for the Hamiltonian of eq. (19) can be written as a set of second-order differential equation for the coupled Harmonic oscillators.

$$\begin{pmatrix} \Omega_e^2 & 0 & 0 & c_1 \Omega_e & 0 & 0 & 2c_1 & 0 \\ 0 & \Omega_e^2 & -c_1 \Omega_e & 0 & 0 & 0 & 0 & 2c_1 \\ 0 & -c_2 & v^2 & 0 & -2c_1 & 0 & 0 & 0 \\ c_2 & 0 & 0 & v^2 & 0 & -2c_1 & 0 & 0 \\ 0 & 0 & -2c_2 \Omega_e & 0 & \Omega_e^2 & 0 & 0 & c_2 \\ 0 & 0 & 0 & -2c_2 \Omega_e & 0 & \Omega_e^2 & -c_2 & 0 \\ 2c_2 \Omega_e & 0 & 0 & -4c_1 c_2 & 0 & c_1 \Omega_e & v^2 & 0 \\ 0 & 2c_2 \Omega_e & -4c_1 c_2 & 0 & c_1 \Omega_e & 0 & 0 & v^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \\ Q_x \\ Q_y \\ P_x \\ P_y \end{pmatrix} = \lambda^2 \begin{pmatrix} x \\ y \\ p_x \\ p_y \\ Q_x \\ Q_y \\ P_x \\ P_y \end{pmatrix}. \quad (20)$$

The Hamiltonian in eq. (10) can now be written completely in terms of the annihilation and creation operators

$$H = \hbar\Omega_e \left(b^+ b + \frac{1}{2} \right) + \hbar v \left(a^+ a + \frac{1}{2} \right) - i e \hbar \left(\frac{\pi \Omega_e}{vmV} \right)^{1/2} [(ab^+ - a^+ b) + (a^+ b^+ - ab)]. \quad (21)$$

The first part in the coupling term is energy-conserving term and play a role in the calculation of the transition probability, whereas the second part does not conserve energy, thus in the electric dipole approximation the second part does not contribute to the transition probability. For simplicity, we assume that an electron has zero z-momentum ($K=0$) and in electric dipole approximation, the quantum number l does not make any difference in the calculation of transition probability. Firstly we calculate the transition probability per unit

time for one photon emission

$$T^e = \frac{2\pi}{\hbar} |\langle \eta_{k+1}, n' | H_1 | \eta_k, n \rangle|^2 \delta(E' - E) \quad (22)$$

where

$$H_1 = -i e \hbar \left(\frac{\pi \Omega_e}{vmV} \right)^{1/2} [(ab^+ - a^+ b)],$$

and

$$E' = n' \hbar \Omega_e + \hbar v, \quad E = n \hbar \Omega_e.$$

Next the transition probability per unit time for one photon absorption can be calculated likewise,

$$T^a = \frac{2\pi}{\hbar} |\langle \eta_k - 1, n' | H_I | \eta_k, n \rangle|^2 \delta(E' - E) \quad (23)$$

where

$$E = n\hbar\Omega_e + \hbar\nu, \quad E' = n'\hbar\Omega_e.$$

We assume that the energy distribution for harmonic oscillator is Boltzmann function

$$f(E) = \frac{2}{\sqrt{\pi}} \left(\frac{1}{kT} \right)^{3/2} E^{1/2} e^{-E/kT}. \quad (24)$$

Then the emission rate can be formulated by averaging initial states

$$\varepsilon = \int \frac{f(E) T^e n_e}{\eta_k V} dE. \quad (25)$$

We then obtain

$$\varepsilon = \frac{3\pi}{4\nu} \omega_p^2 \left(\frac{kT}{\Omega_e \hbar} \right)^{1/2}, \quad (26)$$

where ω_p is the electron plasma frequency. The absorption rate, α can be obtained in a similar way. The net absorption coefficient, $\tilde{\alpha}$ for this radiation is given by

$$\tilde{\alpha} = \frac{\alpha - \varepsilon}{c} = \frac{e^{h\Omega_e/kT} - 1}{c} \varepsilon, \quad (27)$$

where c is the speed of light.

For $\hbar\nu \ll kT$,

$$\tilde{\alpha} = \frac{3\pi\omega_p^2}{4c\nu} \left(\frac{kT}{\hbar\Omega_e} \right)^{-1/2}. \quad (28)$$

Note that net absorption coefficient has the dependence of temperature $T^{-1/2}$, wave frequency $\nu^{-(1/2)}$ (since $\nu \approx \Omega_e$), and electron density n^E , and in accord with previous results¹⁵⁾ which deals with the arbitrary K case.

It should be noted that the diamagnetic term of the Hamiltonian gives rise to a change in frequency from ω to $\nu = \sqrt{\omega(\omega + \gamma)}$, therefore, in ECRH heating the input wave frequency should be $\omega = -2\gamma + \sqrt{\Omega_e^2 + 4\gamma^2}$.

The energy non-conserving term in eq. (21) for the linearly polarized em field can be written as

$$H_I^{RWA} = e\hbar \left(\frac{\pi\Omega_e}{\nu m V} \right)^{1/2} [2i(a^+ b_x - a b_x^+) - (a^+ b_y + a b_y^+)]. \quad (29)$$

For the circularly polarized em field, the RWA term in the interaction Hamiltonian is written as

$$H_I^{RWA} = -2ie\hbar \left(\frac{\pi\Omega_e}{\nu m V} \right)^{1/2} (b_- a^+ - b^+ a_-). \quad (30)$$

The effect of RWA terms on the absorption and emission rate can be explored by estimating the transition probability per unit time (TPPUT). The transition probability per unit time can be calculated using the T-matrix formalism such as in eqs. (22), (23). The calculation will be carried out in subsequent papers.

§3. Conclusions

In this study we focus on the problem of the interaction of a free electron in a uniform magnetic field with a single-mode fully quantized electromagnetic field. We describe the Hamiltonian for two representative polarization of the incident em wave. From these we formulate the equations of motion for the coupled harmonic oscillators, and suggest an approach to obtain the normal mode frequency of vibrations. Then we calculate the net absorption rate of em wave into the free electron gas using the T-matrix formalism. The interaction Hamiltonian has energy non-conserving RWA terms. These RWA terms are manifested in operators forms, which can be utilized to carry out the calculation of the matrix element of T-matrix. The detailed calculation of TPUT including the RWA terms and the evaluation of the eigenvalues λ to obtain the normal mode frequency for circularly polarized em wave will be carried out in subsequent papers. There, we will investigate the effect of RWA terms on the normal mode frequency, and formulate the absorption rate of em wave into the electron system.

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